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3rd DEGREE PARABOLA
AND OPTIMUM RAILWAY TRANSITION CURVES
OF 9th AND 11th DEGREE FOR THEIR
DIFFERENT LENGTHS

Abstract: In this article, due to the common use the 3rd degree parabola transition curve, the authors focused on showing in which range of curves lengths \( l_0 \) the superiority of transition curves of 9th and 11th degree with maximum number of terms persists over the 3rd degree parabola. The measure of evaluation of the curves were values of the objective function (QF) for different lengths \( l_0 \). The authors used one criterion of optimization, namely the integral of absolute value of the acceleration of vehicle body's mass centre along the route. QF values were determined using advanced computer software. Interesting values of QF were obtained assuming the length \( l_0 \) once for the 3rd degree parabola, and once for four considered optimum curves of high degrees.

Keywords: polynomial railway transition curves, computer simulation, optimization

1. INTRODUCTION

The authors of this article for years study the dynamical properties of railway transition curves (TCs) ([4] - [10]). In works [7], [8] and [9] they revealed very good dynamical properties of TCs of 9th and 11th degree with the maximum number of their terms. These features, obtained both for the maximum admissible velocity of vehicle in the circular arc and a lower velocity (guaranteeing ideally balanced lateral interactions on the body), led the authors to investigate the range of lengths in which these properties are valid. The study was made for curves of 9th (defined by the formulas (5) and (6)), and curves of 11th degree (defined by formulas (7) and (8)). Mentioned features show a superiority of TC of 9th and 11th degree, over the 3rd degree parabola.

In this study authors focus on the range of the lengths of transition curves in which mentioned optimum curves are better than the 3rd degree parabola. Measure of quality are the values of the objective function (QF) for different lengths \( l_0 \). Conventional criterion was used - the integral of the absolute value of the acceleration of the body's mass centre...
along the route. QF values were determined using advanced computer software - the vehicle simulation model that takes into account the vehicle-track and vehicle-passenger interactions. Interesting values of quality functions were obtained for different lengths $l_0$ the 3rd degree parabola transition curve, and for the transition curves (5) - (8).

2. VEHICLE MODEL

In the study one model of wagon was used. The model represents 2-axle freight car of the average values of parameters. Despite the virtual character this is a typical 2-axle freight car. It is considered in loaded state. Relatively simple construction of vehicle gives acceptable computation times, which is preferred in basic research and facilitates the interpretation of results. Its structure is shown in Fig. 1c. It is supplemented with discrete models of vertically and laterally flexible track shown in Fig 1a and 1b, respectively. Linearity of the vehicle suspension was assumed. So, linear stiffness and damping elements in vehicle suspension were applied. The same concerns the track models. Here, also linear stiffness and damping elements were applied. One can find all parameters of the model used in detail in [8].

Vehicle model is equipped with a pair of wheel/rail profiles that correspond to the real ones. That is a pair of the nominal (i.e. unworn) S1002/UIC60 profiles that are used all over the Europe. Non-linear geometry of this pair is introduced into the model in a form of table with the contact parameters. In order to calculate non-linear tangential contact forces between wheel and rail well known FASTSIM program by J.J. Kalker was applied. Normal forces in the contact are not constant but influenced by both the geometry and the dynamical effects that make value of a wheelset vertical load variable.

![Fig 1. System's nominal model: (a) track vertically, (b) track laterally, (c) vehicle](image)

The route of interest is characterised in the method by shape of the track centre line which is the general space (3-dimensional) curve. In railway systems such 3-dimensional objects are TCs with their superelevation ramps. A necessary condition to apply the
method is description of the curves (sections) by parametric equations, with the curve's current length \( l \) as the parameter.

### 3. ADOPTED TRANSITION CURVES

The authors of the article studied the polynomial transition curves (TCs) of odd degree 9th and 11th. The parameters of the transition curve are presented in the form of the following equations:

\[
y = \frac{1}{R} \left( \frac{A_1 l^n}{l_0^{n-2}} + \frac{A_{n-1} l^{n-1}}{l_0^{n-3}} + \frac{A_{n-2} l^{n-2}}{l_0^{n-4}} + \frac{A_{n-3} l^{n-3}}{l_0^{n-5}} + \ldots + \frac{A_1 l^0}{l_0^{n}} + \frac{A_2 l^1}{l_0^{n+1}} \right),
\]

\[
k = \frac{d^2 y}{dl^2} = \frac{1}{R} \left[ n(n-1)^2 \frac{A_1 l^{n-2}}{l_0^{n-2}} + (n-1)(n-2) \frac{A_{n-1} l^{n-3}}{l_0^{n-3}} + \ldots + 3 \cdot 2 A_1 l^1 \right],
\]

\[
h = H \left[ n(n-1)^2 \frac{A_1 l^{n-2}}{l_0^{n-2}} + (n-1)(n-2) \frac{A_{n-1} l^{n-3}}{l_0^{n-3}} + \ldots + 4 \cdot 3 A_1 l^1 \right],
\]

\[
i = \frac{dh}{dl} = H \left[ n(n-1)(n-2) \frac{A_1 l^{n-1}}{l_0^{n-2}} + (n-1)(n-2)(n-3) \frac{A_{n-1} l^{n-2}}{l_0^{n-3}} + \ldots + 5 \cdot 4 \cdot 3 A_1 l^2 + 4 \cdot 3 \cdot 2 A_1 l^1 \right],
\]

where \( y, k, h, \) and \( i \) define curve lateral co-ordinate, curvature, superelevation, and inclination of superelevation ramp, respectively. The \( R, H, l_0, \) and \( l \) define curve minimum radius (at its end), maximum superelevation (at the curve end), total curve length, and curve current length, respectively. The \( A_i \) are polynomial coefficients (\( i = n, n-1, \ldots, 4, 3 \)) where \( n \) is polynomial degree. Number of the polynomial terms (terms in Eqs. (1)-(4)) must not be smaller than 2. On the other hand the smallest degree \( n_{min} \) of the last term in Eq. (1) must be \( n_{min} \geq 3 \). Such definition of the curves gives possibility of proper values \( k \) and \( h \) at TC’s terminal points. They should be equal to 0 at the initial points and \( 1/R \) and \( H \) at the end points. Note, that values for both ones always are equal to 0 for \( l=0 \). In order to ensure values \( 1/R \) and \( H \) for \( l=L \), normalisation of the coefficients is necessary ([1]). In this work, condition on tangency of \( k, h \) and \( i \) was also imposed: i.e. \( k'(0)=0, k'(l_0)=0, i'(0)=0 \) and \( i'(l_0)=0 \). So each time if authors say about a tangency of \( k, h \) and \( i \), they mean tangency of these quantities to the courses of these functions in contiguous straight track (ST) and circular arc (CC).
Coefficients $A'_i$ are obtained which satisfy constraints imposed on their values. The problem just formulated is a classical formulation of the static constrained optimisation. It is solved with the library procedure that utilises moving penalty function algorithm combined with Powell's method of conjugate directions.

4. OPTIMUM TRANSITION CURVES OF 9th AND 11th DEGREE TAKEN TO THE ANALYSIS

In [7], [8] and [9] it is shown that the best dynamical properties have transition curves of 9th and 11th degree with the maximum number of terms - (5) - (8). Curvature of these curves do not have a tangency in the terminal points of the curve. Curvature of these curves are shown in Fig 2.

For each polynomial transition curve (both odd and even degree) geometrical demands were imposed that one wants or does not want to take into account. Possible combinations of the coefficients for those demands are shown in Tab 1. Generally, in this paper authors focused only on polynomial curves of 9th and 11th degree with the maximum number of terms.

### Possible polynomial configurations for different geometrical demands

<table>
<thead>
<tr>
<th>Type of demand</th>
<th>Demand IDZ=1 (proper values of $r$ and $h$ in TCs' terminal points)</th>
<th>Demand IDZ=2 (tangence of $r$ and $h$ functions at TCs' terminal points)</th>
<th>Demand IDZ=3 (tangence of $h$ slope, i.e. of $i$, at TCs' terminal points)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial degree (terms number in the initial polyn.)</td>
<td>Number of terms</td>
<td>Number of terms</td>
<td>Number of terms</td>
</tr>
<tr>
<td>5th degree (IW=2)</td>
<td>IW=2; IW=3</td>
<td>IW=2 (single curve)</td>
<td>Not to be satisfied</td>
</tr>
<tr>
<td>7th degree (IW=3)</td>
<td>IW=2; ...; IW=5</td>
<td>IW=2 (single curve); ...; IW=4</td>
<td>IW=3 (single curve)</td>
</tr>
<tr>
<td>9th degree (IW=4)</td>
<td>IW=2; ...; IW=7</td>
<td>IW=2 (single curve); ...; IW=6</td>
<td>IW=3 (single curve); ...; IW=5</td>
</tr>
<tr>
<td>11th degree (IW=5)</td>
<td>IW=2; ...; IW=9</td>
<td>IW=2 (single curve); ...; IW=8</td>
<td>IW=3 (single curve); ...; IW=7</td>
</tr>
</tbody>
</table>

In [7] and [8] the authors assumed the following conditions of optimization:

- velocity of wheel vertical rise along the superelevation ramp $f = 56$ mm/s,
- curve radius $R = 600$ m, cant $H = 0.15$ m, vehicle velocity $\nu = 24.26$ m/s (guarantees ideal balance between transversal components of gravity and centrifugal forces) and $\nu = 30.79$ m/s (represents maximum admissible vehicle velocity in the curved track),
- length of transition curve $l_0$ calculated according with [1],
- criterion – minimization of integral of the absolute value of the acceleration of the body's mass centre along the route.
and they received formulas for 4 optimum transition curves (5)-(8).

These obtained formulas are:

a) 9th degree, length of transition curve $l_0=142.15$ m, unbalanced lateral acceleration in the curved track $a=0$ m/s$^2$, vehicle velocity $v=24.26$ m/s:

$$y = \frac{l}{600} \left( -0.0994164 \cdot \left( \frac{l^9}{142.15^5} \right) + 0.44678 \cdot \left( \frac{l^8}{142.15^6} \right) - 0.715041 \cdot \left( \frac{l^7}{142.15^7} \right) 
+ 0.417056 \cdot \left( \frac{l^6}{142.15^8} \right) + 0.000226491 \cdot \left( \frac{l^5}{142.15^9} \right) + 0.00638498 \cdot \left( \frac{l^4}{142.15^{10}} \right) 
+ 0.0968742 \cdot \left( \frac{l^3}{142.15^{11}} \right) \right) \right) \right) \right)$$

(5)

b) 9th degree, length of transition curve $l_0=180.46$ m, unbalanced lateral acceleration in the curved track $a=0.6$ m/s$^2$, vehicle velocity $v=30.79$ m/s:

$$y = \frac{l}{600} \left( -0.220519 \cdot \left( \frac{l^9}{180.46^5} \right) + 0.992337 \cdot \left( \frac{l^8}{180.46^6} \right) - 1.58774 \cdot \left( \frac{l^7}{180.46^7} \right) 
+ 0.926174 \cdot \left( \frac{l^6}{180.46^8} \right) + 0 \cdot \left( \frac{l^5}{180.46^9} \right) + 0 \cdot \left( \frac{l^4}{180.46^{10}} \right) + 0.0343937 \cdot \left( \frac{l^3}{180.46^{11}} \right) \right) \right) \right) \right)$$

(6)

c) 11th degree, length of transition curve $l_0=159.86$ m, unbalanced lateral acceleration in the curved track $a=0$ m/s$^2$, vehicle velocity $v=24.26$ m/s:

$$y = \frac{l}{600} \left( 0.166895 \cdot \left( \frac{l^{11}}{159.86^5} \right) - 0.917482 \cdot \left( \frac{l^{10}}{159.86^6} \right) + 1.96613 \cdot \left( \frac{l^9}{159.86^7} \right) 
- 1.96607 \cdot \left( \frac{l^8}{159.86^8} \right) + 0.78519 \cdot \left( \frac{l^7}{159.86^9} \right) - 0.000593593 \cdot \left( \frac{l^6}{159.86^{10}} \right) 
+ 0.0000904 \cdot \left( \frac{l^5}{159.86^{11}} \right) + 0.0119314 \cdot \left( \frac{l^4}{159.86^{12}} \right) + 0.151419 \cdot \left( \frac{l^3}{159.86^{13}} \right) \right) \right) \right)$$

(7)

d) 11th degree, length of transition curve $l_0=202.94$ m, unbalanced lateral acceleration in the curved track $a=0.6$ m/s$^2$, vehicle velocity $v=30.79$ m/s:
y = \frac{1}{600} \left( 0.527815 \cdot \left(\frac{l^{11}}{202.94^9}\right) - 2.90298 \cdot \left(\frac{l^{10}}{202.94^8}\right) + 6.22068 \cdot \left(\frac{l^9}{202.94^7}\right) \right.
\left. - 6.22068 \cdot \left(\frac{l^8}{202.94^6}\right) + 2.48827 \cdot \left(\frac{l^7}{202.94^5}\right) + 0 \cdot \left(\frac{l^6}{202.94^4}\right) \right)
\left. + 0.000129549 \cdot \left(\frac{l^5}{202.94^3}\right) + 0.00163316 \cdot \left(\frac{l^4}{202.94^2}\right) + 0.0247312 \cdot \frac{l^3}{202.94} \right) \tag{8}

The curves – (5)-(8) – found in the works [7], [8] have properties better than both the 3rd degree parabola and so called standard curves, if we take into account QF assumed. Authors managed to find new formulas for such curves not found by them in the literature available to them. As mentioned earlier the normalised curvatures (superelavation ramps) of these curves – being something between standard curves and 3rd degree parabola – are presented in Fig. 2. They have characteristic bends in first and last point of curve. It is indicated by the arrows. Here, length of TC was assumed to be equal to 100 m.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{Normalised curvatures (superelevation ramps) for TC of 9th and 11th degree ([7], [8]), arrows indicate curvatures' (superelevation ramps') bends in extreme points}
\end{figure}

The results of the study explain, why engineers didn’t use polynomial transition curves successfully. Authors of the article know, that these curves aren’t popular in use. Engineers used polynomial curves of the lower degrees, which curvatures (and superelevation ramps) had tangency in extreme points – e.g. Bloss curve. New shapes of railway transition curve obtained by the authors of the article suggest need for further research of transition curves and set new direction of the research. Despite the fact, that new formulas for optimum transition curves are defined, one can also look wider at the results received. The new shapes indicate, that scientists and engineers should focus on curves, which combine
features of 3rd degree parabola and standard polynomial curves of higher degrees ([7], [8]). Transitions curves, which were presented by authors of [7] and [8] are better than 3rd degree parabola for most acceptable lengths of curves – point no. 5 of current article.

5. RESULTS

The results of this analysis are shown in 8 pictures. Values of quality function for transition curves of 9th degree are presented in Fig. 3a and 3b and values of quality function for transition curves of 11th degree are presented in Fig. 4a and 4b. Figures 3a and 4a correspond to the vehicle velocity guaranteeing an ideal balance between transversal components of gravity and centrifugal forces. Figures 3b and 4b correspond to the maximum admissible vehicle velocity in the curved track. The values of these velocities are respectively equal to 24.26 m/s and 30.79 m/s. Details of the simulation conditions are in accordance with the conditions specified in point no 4 - point a) for Fig. 3a, point b) for Fig. 3b, point c) for Fig. 4a and point d) for Fig. 4b.

Analysis of the results in Figs. 3a, 3b, 4a and 4b have qualitative similarity. In all Figs. x-axis this is a length of transition curves and y-axis this are values of quality function for a given curve and a given length. In all Figs. there is a point of intersection of the curve for 3rd degree parabola and the curves defined by formulas (5)-(8). These points of intersection correspond the following lengths of the curve, respectively:

a) 93 m - 9th degree, 24.26 m/s,

b) 158 m - 9th degree, 30.79 m/s,

c) 114 m - 11th degree, 24.26 m/s,

d) 159 m - 11th degree, 30.79 m/s.

All the Figs. show the superiority of optimum transition curves of 9th and 11th of degree ((5)-(8)) over 3rd degree parabola above the specified length (a point of intersection). The mentioned lengths are obviously smaller than the minimum lengths of the optimum curve \( l_o \) [7], which are indicated in Figs. 3 and 4 by vertical lines. The corresponding minimum values are 142.15, 180.46, 159.86 and 202.94 m. The minimum length of the 3rd degree parabola for velocity \( v=24.26 \) m/s and 30.79 m/s are respectively 129.97 m and 164.99 m [7]. So for all acceptable lengths of transition curves, the curves (5)-(8) are better than 3rd degree parabola. 3rd degree parabola is better only for lengths lower than 93, 158, 114 and 159 m. These lengths of the curves are however not acceptable, because they are lower than minimum values of lengths - 142.15, 180.46, 159.86 and 202.94 m

Figs. 5 and 6 illustrate a comparison of the dynamics characteristics - displacements and accelerations of the vehicle body mass centre - for the curve (8) and the 3rd degree parabola. In the first case, the length of curves - 3rd degree parabola and (8) - was assumed to be equal to 120 m. Here, parabola 3rd degree has a superiority over the curve (8). In the second case - for a length 180 m for both curves - the curve (8) dominates over the 3rd degree parabola. All these lengths are not acceptable, because they are lower than minimum length \( l_o \) – 202.94 m.
Fig 3. Comparison of values of QF for different TCs lengths: a) curve (5) and 3rd degree parabola, b) curve (6) and 3rd degree parabola

Fig 4. Comparison of values of QF for different TCs lengths: a) curve (7) and 3rd degree parabola, b) curve (8) and 3rd degree parabola

Fig 5. Results of simulation for vehicle body for the curve (8) – length 120 m – and 3rd degree parabola: a) lateral displacement, b) lateral acceleration
Fig 6. Results of simulation for vehicle body for the curve (8) – length 180 m – and 3rd degree parabola: a) lateral displacement, b) lateral acceleration

4. CONCLUSIONS

The study shows the ranges of lengths of transition curves for whom the optimum transition curves of 9th and 11th degrees (with curvatures’ bends in terminal points) are more favorable than the most used parabola 3rd degree parabola. The objective of future research of authors’ article is to clarify, why the curvatures’ bends of railway transition curves at the terminal points of the curve - the beginning and the end - have a relatively small negative impact on vehicle dynamics while running on such curves.

The authors puts forward and is going to explain the following research hypotheses. These are hypotheses, that can possibly explain the problem, that the curvatures’ bends of railway transition curves obtained as results of the author’s earlier research have a relatively small negative impact on vehicle dynamics. These hypotheses are:

a) beneficial effect on the body’s response to curvature’s bend can have a selection of vehicle suspension system;
b) is worth noting, that the shape of the curvature function does not map trajectory of the vehicle in the plan. It is mapped by y coordinate. So, bend in curvature, as opposed to a bend in superelevation ramp, does not cause direct bend in a trajectory. This quantity can affect the dynamic behaviour, but it seems to be smaller than in the case of bends directly in the trajectory;
c) inclusion in the analysis the vertical dynamics (objective functions) can change both the assessment of the curves and the results of the optimisation;
d) relatively mild motion conditions in the TC adopted because of the regulations should be changed for reasons of research for more severe;
e) modification objective function calculation so that initial and end zones have bigger weights (importance) than the middle zone can also have beneficial effect on the body’s response. It is possible that bigger length of the middle zone causes that shape of the terminal zones has become less important.

References


PARABOLA 3. STOPNIA A OPTYMALNE KOLEJOWE KRZYWE PRZEJŚCIOWE STOPNIA 9. ORAZ 11. DLA ICH RÓŻNYCH DŁUGOŚCI

Streszczenie: W artykule, ze względu na powszechność stosowania krzywej parabolicznej 3. stopnia, skupiono się na pokazaniu, w jakim zakresie długości krzywych \(l_0\) utrzymuje się przewaga krzywych przejściowych 9. i 11. stopnia z maksymalną liczbą wyrazów nad parabolą 3. stopnia. Miarą oceny własności krzywych są wartości funkcji celu (FC) dla różnych długości \(l_0\). Wykorzystane zostało kryterium znormalizowanej całki z wartości bezwzględnej przyspieszenia środka masy nadwozia po długości drogi. Wartości FC wyznaczono za pomocą zaawansowanego oprogramowania komputerowego. Interesujące wartości FC uzyskano przyjmując dla badanej długości \(l_0\) raz parabolę 3. stopnia, a raz jedną z czterech rozważanych krzywych optymalnych wysokich stopni.

Słowa kluczowe: wielomianowe krzywe przejściowe, symulacja komputerowa, optymalizacja